# Calculation of estimates of principal measurand for examples 1A and 1B

## Contents



## 1 Introduction

This document describes the information that must be provided by the user to define the sensor responses to the principal and subsidiary measurands in the examples contained in MATLAB scripts `DataFusionSoftware_1A.m` and `DataFusionSoftware_1B.m`.

In particular, section 2 contains a list of all the numerical input that the user is expected to provide. For users who are interested, section 3 lists previously calculated information that is used when determining the sensor responses, while sections 4 and 5 provide full mathematical descriptions of how the sensor response values and estimates of the principal measurand (and associated uncertainties) are calculated, respectively, by the function `LinearCalibrationInterdependenceCov.m`.

## 2 User input

The responses of the sensors to the principal measurand are defined by the following information provided by the user:

### Sampling and quantization

- $f_2^{(1)}$, the sampling frequency (in Hz) [cell B10],
- (optional) $n_{\mathrm{B}}^{(1)}$, the number of bits for quantization [cell B11],
- (optional) $s^{(1)}$, the saturation value [cell B12].

It is assumed that $f_2^{(1)} < f_1^{(1)}$, i.e., the sampling frequency for the sensors is less than the sampling frequency used when calculating the 'true' principal measurand (section 3).

**Behaviour of sensors**

- $n_{\rm S}^{(1)}$, the number of sensors [cell B9 – filled in automatically from information in column C],
- $A_{l_{\rm S}}^{(1)},\ u_{\rm r}\left(A_{l_{\rm S}}^{(1)}\right),\ l_{\rm S}=1,\ldots,n_{\rm S}^{(1)}$, the estimates of the offset parameter and their associated relative standard uncertainties [columns D and G],
- $B_{l_{\rm S}}^{(1)},\ u_{\rm r}\left(B_{l_{\rm S}}^{(1)}\right),\ l_{\rm S}=1,\ldots,n_{\rm S}^{(1)}$, the estimates of the gain parameter for the subsidiary measurand and their associated relative standard uncertainties [columns E and H],
- $C_{l_{\rm S}}^{(1)},\ u_{\rm r}\left(C_{l_{\rm S}}^{(1)}\right),\ l_{\rm S}=1,\ldots,n_{\rm S}^{(1)}$, the estimates of the gain parameter for the principal measurand and their associated relative standard uncertainties [columns F and I],
- $\mathrm{cov}_r\left(A_{l_{\rm S}}^{(1)},B_{l_{\rm S}}^{(1)}\right),\mathrm{cov}_r\left(A_{l_2}^{(1)},C_{l_{\rm S}}^{(1)}\right),\ \mathrm{cov}_r\left(B_{l_2}^{(1)},C_{l_{\rm S}}^{(1)}\right),\ l_{\rm S}=1,\ldots,n_{\rm S}^{(1)}$, the relative covariances associated with the estimates of the offset and gain parameters [columns J, K and L],
- $\delta_{l_{\rm S}}^{(1)},\ l_{\rm S}=1,\ldots,n_{\rm S}^{(1)}$, the levels of additive noise in the sensor output [column M].

**Behaviour of faulty sensors**

- $n_{\rm F}^{(1)}$, the number of faulty sensors [cell P9 – filled in automatically from information in column Q],
- $I_{{\rm F},l_{\rm F}}^{(1)},\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the indices of the faulty sensors [column Q],
- $D_{l_{\rm F}}^{(1)},\ u_{\rm r}\left(D_{l_{\rm F}}^{(1)}\right),\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the estimates of the offset parameter and their associated relative standard uncertainties [columns R and U],
- $E_{l_{\rm F}}^{(1)},\ u_{\rm r}\left(E_{l_{\rm F}}^{(1)}\right),\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the estimates of the gain parameter for the subsidiary measurand and their associated relative standard uncertainties [columns S and V],
- $F_{l_{\rm F}}^{(1)},\ u_{\rm r}\left(F_{l_{\rm F}}^{(1)}\right),\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the estimates of the gain parameter for the principal measurand and their associated relative standard uncertainties [columns T and W],
- $\mathrm{cov}_r\left(D_{l_{\rm F}}^{(1)},E_{l_{\rm F}}^{(1)}\right),\ \mathrm{cov}_r\left(D_{l_{\rm F}}^{(1)},F_{l_{\rm F}}^{(1)}\right),\ \mathrm{cov}_r\left(E_{l_{\rm F}}^{(1)},F_{l_{\rm F}}^{(1)}\right),\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the relative covariances associated with the estimates of the offset and gain parameters [columns X, Y and Z],
- $T_{{\rm F},l_{\rm F}}^{(1),1},\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the times at which sensors begin behaving faultily [column AA],
- $T_{{\rm F},l_{\rm F}}^{(1),2},\ l_{\rm F}=1,\ldots,n_{\rm F}^{(1)}$, the times at which sensors end behaving faultily [column AB].

**Missing data**

- $n_{\mathrm{M}}^{(1)}$, the number of sensors for which packets of data are missing [cell AE9 – filled in automatically from information in column AF],
- $n_{\mathrm{D}}^{(1)}$, the number of data points in each packet [cell AE10],
- $I_{\mathrm{M},l_{\mathrm{M}}}^{(1)}$, $l_{\mathrm{M}} = 1, \ldots, n_{\mathrm{M}}^{(1)}$, the indices of the sensors for which packets of data are missing [column AF],
- $p_{\mathrm{M},l_{\mathrm{M}}}^{(1)}$, $l_{\mathrm{M}} = 1, \ldots, n_{\mathrm{M}}^{(1)}$, the proportion (expressed as a percentage) of the data packets that are missing within the assigned time interval [column AG],
- $T_{\mathrm{M},l_{\mathrm{M}}}^{(1),1}$, $l_{\mathrm{M}} = 1, \ldots, n_{\mathrm{M}}^{(1)}$, the times at which sensors begin missing packets of data [column AH],
- $T_{\mathrm{M},l_{\mathrm{M}}}^{(1),2}$, $l_{\mathrm{M}} = 1, \ldots, n_{\mathrm{M}}^{(1)}$, the times at which sensors cease missing packets of data [column AI].

# 3 Additional input

The following information, previously calculated, is used:

- $t_{1,i_{11}}^{(1)}$, $i_{11} = 1, \ldots, m_1^{(1)}$, the times at which the ‘true’ values of the principal measurand are calculated (corresponding to the sampling frequency $f_1^{(1)}$),
- $y_{1,i_{11}}^{(1)}$, $i_{11} = 1, \ldots, m_1^{(1)}$, the ‘true’ values of the principal measurand, stored in vector $\mathbf{y}_1^{(1)}$,
- $t_{1,i_{21}}^{(2)}$, $i_{21} = 1, \ldots, m_1^{(2)}$, the times at which the ‘true’ values of the subsidiary measurand are calculated (corresponding to the sampling frequency $f_1^{(2)}$),
- $y_{1,i_{21}}^{(2)}$, $i_{21} = 1, \ldots, m_1^{(2)}$, the ‘true’ values of the subsidiary measurand, stored in vector $\mathbf{y}_1^{(2)}$,
- $t_{2,i_{22}}^{(2)}$, $i_{22} = 1, \ldots, m_2^{(2)}$, the times at which estimates of the values of the subsidiary measurand are calculated (corresponding to the sampling frequency $f_2^{(2)}$),
- $\widehat{y}_{2,i_{22}}^{(2)}$, $i_{22} = 1, \ldots, m_2^{(2)}$, the estimates of the values of the subsidiary measurand, stored in vector $\widehat{\mathbf{y}}_2^{(2)}$,
- $u\left(\widehat{y}_{2,i_{22}}^{(2)}\right)$, $i_{22} = 1, \ldots, m_2^{(2)}$, the uncertainties associated with the estimates of the values of the subsidiary measurand.

## 4 Sensor response values

The array $\widetilde{\mathbf{V}}_2^{(1)}$ of sensor response values is given by

$$\widetilde{\mathbf{V}}_2^{(1)} = \begin{bmatrix} \widetilde{\mathbf{v}}_{2,1}^{(1)} & \ldots & \widetilde{\mathbf{v}}_{2,n_\mathrm{S}^{(1)}}^{(1)} \end{bmatrix},$$

where

$$\widetilde{\mathbf{v}}_{2,l_\mathrm{S}}^{(1)} = \begin{bmatrix} \widetilde{v}_{2,l_\mathrm{S},1}^{(1)} \\ \vdots \\ \widetilde{v}_{2,l_\mathrm{S},m_2^{(1)}}^{(1)} \end{bmatrix}$$

is the vector of response values for sensor $l_\mathrm{S}$ and is obtained as follows:

1. Evaluate the 'true' subsidiary measurand values $\widetilde{y}_{1,i_{11}}^{(2)}$ corresponding to the times $t_{1,i_{11}}^{(1)}$, $i_{11} = 1, \ldots, m_1^{(1)}$, by applying linear interpolation to the subsidiary measurand values $y_{1,i_{21}}^{(2)}$ corresponding to the times $t_{1,i_{21}}^{(1)}$, $i_{21} = 1, \ldots, m_1^{(2)}$.
2. Evaluate the vector of sensor responses

$$\mathbf{v}_{1,l_\mathrm{S}}^{(1)} = A_{l_\mathrm{S}}^{(1)*} + B_{l_\mathrm{S}}^{(1)*}\widetilde{\mathbf{y}}_1^{(2)} + C_{l_\mathrm{S}}^{(1)*}\mathbf{y}_1^{(1)} + \mathbf{r}_{l_\mathrm{S}}^{(1)},$$

corresponding to the times $t_{1,i_{11}}^{(1)}$, $i_{11} = 1, \ldots, m_1^{(1)}$, where

$$\widetilde{\mathbf{y}}_1^{(2)} = \begin{bmatrix} \widetilde{y}_{1,1}^{(2)} \\ \vdots \\ \widetilde{y}_{1,m_1^{(1)}}^{(2)} \end{bmatrix},$$

$$\begin{bmatrix} A_{l_\mathrm{S}}^{(1)*} \\ B_{l_\mathrm{S}}^{(1)*} \\ C_{l_\mathrm{S}}^{(1)*} \end{bmatrix} \sim \mathrm{N}\left( \begin{bmatrix} A_{l_\mathrm{S}}^{(1)} \\ B_{l_\mathrm{S}}^{(1)} \\ C_{l_\mathrm{S}}^{(1)} \end{bmatrix}, \boldsymbol{\Sigma}_{l_\mathrm{S}}^{(1)} \right),$$

with

$$\boldsymbol{\Sigma}_{l_\mathrm{S}}^{(1)} = \begin{bmatrix} \left(\dfrac{u_\mathrm{r}\left(A_{l_\mathrm{S}}^{(1)}\right) A_{l_\mathrm{S}}^{(1)}}{100}\right)^2 & \dfrac{\mathrm{cov}_\mathrm{r}\left(A_{l_\mathrm{S}}^{(1)}, B_{l_\mathrm{S}}^{(1)}\right) A_{l_\mathrm{S}}^{(1)} B_{l_\mathrm{S}}^{(1)}}{100} & \dfrac{\mathrm{cov}_\mathrm{r}\left(A_{l_\mathrm{S}}^{(1)}, C_{l_\mathrm{S}}^{(1)}\right) A_{l_\mathrm{S}}^{(1)} C_{l_\mathrm{S}}^{(1)}}{100} \\ \dfrac{\mathrm{cov}_\mathrm{r}\left(A_{l_\mathrm{S}}^{(1)}, B_{l_\mathrm{S}}^{(1)}\right) A_{l_\mathrm{S}}^{(1)} B_{l_\mathrm{S}}^{(1)}}{100} & \left(\dfrac{u_\mathrm{r}\left(B_{l_\mathrm{S}}^{(1)}\right) B_{l_\mathrm{S}}^{(1)}}{100}\right)^2 & \dfrac{\mathrm{cov}_\mathrm{r}\left(B_{l_\mathrm{S}}^{(1)}, C_{l_\mathrm{S}}^{(1)}\right) B_{l_\mathrm{S}}^{(1)} C_{l_\mathrm{S}}^{(1)}}{100} \\ \dfrac{\mathrm{cov}_\mathrm{r}\left(A_{l_\mathrm{S}}^{(1)}, C_{l_\mathrm{S}}^{(1)}\right) A_{l_\mathrm{S}}^{(1)} C_{l_\mathrm{S}}^{(1)}}{100} & \dfrac{\mathrm{cov}_\mathrm{r}\left(B_{l_\mathrm{S}}^{(1)}, C_{l_\mathrm{S}}^{(1)}\right) B_{l_\mathrm{S}}^{(1)} C_{l_\mathrm{S}}^{(1)}}{100} & \left(\dfrac{u_\mathrm{r}\left(C_{l_\mathrm{S}}^{(1)}\right) C_{l_\mathrm{S}}^{(1)}}{100}\right)^2 \end{bmatrix},$$

and

$$\mathbf{r}_{l_{\mathrm{S}}}^{(1)} = \begin{bmatrix} r_{l_{\mathrm{S}},1}^{(1)} \\ \vdots \\ r_{l_{\mathrm{S}},m_1^{(1)}}^{(1)} \end{bmatrix},$$

with

$$r_{l_{\mathrm{S}},i_{11}} \sim \mathrm{N}\left(0, \left(\delta_{l_{\mathrm{S}}}^{(1)}\right)^2\right),\ i_{11} = 1, \ldots, m_1^{(1)}.$$

3. For sensors that are faulty, the response values within the time intervals $\left[T_{\mathrm{F},l_{\mathrm{F}}}^{(1),1}, T_{\mathrm{F},l_{\mathrm{F}}}^{(1),2}\right]$ are obtained similarly to those in step 2, but using the parameters $D_{l_{\mathrm{F}}}^{(1)}$, $u_{\mathrm{r}}\left(D_{l_{\mathrm{F}}}^{(1)}\right)$, $E_{l_{\mathrm{F}}}^{(1)}$, $u_{\mathrm{r}}\left(E_{l_{\mathrm{F}}}^{(1)}\right)$, $F_{l_{\mathrm{F}}}^{(1)}$, $u_{\mathrm{r}}\left(F_{l_{\mathrm{F}}}^{(1)}\right)$, $\mathrm{cov}_r\left(D_{l_{\mathrm{F}}}^{(1)}, E_{l_{\mathrm{F}}}^{(1)}\right)$, $\mathrm{cov}_r\left(D_{l_{\mathrm{F}}}^{(1)}, F_{l_{\mathrm{F}}}^{(1)}\right)$ and $\mathrm{cov}_r\left(E_{l_{\mathrm{F}}}^{(1)}, F_{l_{\mathrm{F}}}^{(1)}\right)$, $l_{\mathrm{F}} = 1, \ldots, n_{\mathrm{F}}^{(1)}$.

4. Determine the times $t_{2,i_{12}}^{(1)},\ i_{12} = 1, \ldots, m_2^{(1)}$, at which the sensor response values are to be evaluated.

5. Evaluate the sensor response values $v_{2,l_{\mathrm{S}},i_{12}}^{(1)}$ corresponding to the times $t_{2,i_{12}}^{(1)}$, $i_{12} = 1, \ldots, m_2^{(1)}$, by applying linear interpolation to the sensor responses $v_{1,l_{\mathrm{S}},i_{11}}^{(1)}$ corresponding to the times $t_{1,i_{11}}^{(1)},\ i_{11} = 1, \ldots, m_1^{(1)}$.

6. The sensor response values $v_{2,l_{\mathrm{S}},i_{12}}^{(1)},\ i_{12} = 1, \ldots, m_2^{(1)}$, are then quantized according to the values of $n_{\mathrm{B}}^{(1)}$ and $s^{(1)}$ (if present) to give values

$$\widetilde{v}_{2,l_{\mathrm{S}},i_{12}}^{(1)},\ i_{12} = 1, \ldots, m_2^{(1)}.$$

If no values have been provided for $n_{\mathrm{B}}^{(1)}$ and $s^{(1)}$, then

$$\widetilde{v}_{2,l_{\mathrm{S}},i_{12}}^{(1)} = v_{2,l_{\mathrm{S}},i_{12}}^{(1)},\ i_{12} = 1, \ldots, m_2^{(1)}.$$

7. For sensors that have missing data, the proportions $p_{\mathrm{M},l_{\mathrm{M}}}^{(1)}$ of data packets (a data packet is defined to be a group of data points that are sequential in time) chosen randomly within the time intervals $\left[T_{\mathrm{M},l_{\mathrm{M}}}^{(1),1}, T_{\mathrm{M},l_{\mathrm{M}}}^{(1),2}\right]$ have their values set to Not-a-Number (NaN), $l_{\mathrm{M}} = 1, \ldots, n_{\mathrm{M}}^{(1)}$.

# 5 Principal measurand estimates and associated uncertainties

Evaluate estimates $\widetilde{z}^{(2)}_{2,i_{12}}$ of the subsidiary measurand and associated uncertainties $u\left(\widetilde{z}^{(2)}_{2,i_{12}}\right)$ corresponding to the times $t^{(1)}_{2,i_{12}}$, $i_{12} = 1, \ldots, m^{(1)}_2$, by applying linear interpolation to the estimates $\widehat{y}^{(2)}_{2,i_{22}}$ and associated uncertainties $u\left(\widehat{y}^{(2)}_{2,i_{22}}\right)$ corresponding to the times $t^{(2)}_{2,i_{22}}$, $i_{22} = 1, \ldots, m^{(2)}_2$. The array $\mathbf{Y}^{(1)}_2$ of principal measurand estimates is given by

$$\mathbf{Y}^{(1)}_2 = \begin{bmatrix} y^{(1)}_{2,1,1} & \cdots & y^{(1)}_{2,n^{(1)}_{\mathrm{S}},1} \\ \vdots & \ddots & \vdots \\ y^{(1)}_{2,1,m^{(2)}_2} & \cdots & y^{(1)}_{2,n^{(1)}_{\mathrm{S}},m^{(2)}_2} \end{bmatrix},$$

where

$$y^{(1)}_{2,l_{\mathrm{S}},i_{12}} = \frac{\widetilde{v}^{(1)}_{2,l_{\mathrm{S}},i_{12}} - A^{(1)}_{l_{\mathrm{S}}} - B^{(1)}_{l_{\mathrm{S}}} \widetilde{z}^{(2)}_{2,i_{12}}}{C^{(1)}_{l_{\mathrm{S}}}}.$$

At each time value $t^{(1)}_{2,i_{12}}$, the covariance matrix $\mathbf{V}^{(1)}_{i_{12}}$ of dimension $n^{(1)}_{\mathrm{S}} \times n^{(1)}_{\mathrm{S}}$ associated with the estimates $y^{(1)}_{2,l_{\mathrm{S}},i_{12}}$, $l_{\mathrm{S}} = 1, \ldots, n^{(1)}_{\mathrm{S}}$, of the principal measurand has diagonal elements

$$\mathbf{V}^{(1)}_{i_{12}}(l_{\mathrm{S}}, l_{\mathrm{S}}) = \left(\frac{1}{C^{(1)}_{l_{\mathrm{S}}}}\right)^2 \left(\frac{u_{\mathrm{r}}\left(A^{(1)}_{l_{\mathrm{S}}}\right) A^{(1)}_{l_{\mathrm{S}}}}{100}\right)^2 + \left(\frac{\widetilde{z}^{(2)}_{2,i_{12}}}{C^{(1)}_{l_{\mathrm{S}}}}\right)^2 \left(\frac{u_{\mathrm{r}}\left(B^{(1)}_{l_{\mathrm{S}}}\right) B^{(1)}_{l_{\mathrm{S}}}}{100}\right)^2 +$$
$$\left(\frac{y^{(2)}_{2,l_{\mathrm{S}},i_2}}{C^{(1)}_{l_{\mathrm{S}}}}\right)^2 \left(\frac{u_{\mathrm{r}}\left(C^{(1)}_{l_{\mathrm{S}}}\right) C^{(1)}_{l_{\mathrm{S}}}}{100}\right)^2 + 2\left(\frac{\widetilde{z}^{(2)}_{2,i_{12}}}{\left(C^{(1)}_{l_{\mathrm{S}}}\right)^2}\right) \left(\frac{\mathrm{cov}_{\mathrm{r}}\left(A^{(1)}_{l_{\mathrm{S}}}, B^{(1)}_{l_{\mathrm{S}}}\right) A^{(1)}_{l_{\mathrm{S}}} B^{(1)}_{l_{\mathrm{S}}}}{100}\right) +$$
$$2\left(\frac{y^{(2)}_{2,l_{\mathrm{S}},i_2}}{\left(C^{(1)}_{l_{\mathrm{S}}}\right)^2}\right) \left(\frac{\mathrm{cov}_{\mathrm{r}}\left(A^{(1)}_{l_{\mathrm{S}}}, C^{(1)}_{l_{\mathrm{S}}}\right) A^{(1)}_{l_{\mathrm{S}}} C^{(1)}_{l_{\mathrm{S}}}}{100}\right) +$$
$$2\left(\frac{\widetilde{z}^{(2)}_{2,i_{12}} y^{(2)}_{2,l_{\mathrm{S}},i_2}}{\left(C^{(1)}_{l_{\mathrm{S}}}\right)^2}\right) \left(\frac{\mathrm{cov}_{\mathrm{r}}\left(B^{(1)}_{l_{\mathrm{S}}}, C^{(1)}_{l_{\mathrm{S}}}\right) B^{(1)}_{l_{\mathrm{S}}} C^{(1)}_{l_{\mathrm{S}}}}{100}\right) + \left(\frac{\delta^{(1)}_{l_{\mathrm{S}}}}{C^{(1)}_{l_{\mathrm{S}}}}\right)^2 + \left(\frac{B^{(1)}_{l_{\mathrm{S}}}}{C^{(1)}_{l_{\mathrm{S}}}}\right)^2 u^2\left(\widetilde{z}^{(2)}_{2,i_{12}}\right)$$

and off-diagonal elements

$$\mathbf{V}^{(1)}_{i_{12}}(l_{\mathrm{S}}, r_{\mathrm{S}}) = \left(\frac{B^{(1)}_{l_{\mathrm{S}}} B^{(1)}_{r_{\mathrm{S}}}}{C^{(1)}_{l_{\mathrm{S}}} C^{(1)}_{r_{\mathrm{S}}}}\right) u^2\left(\widetilde{z}^{(2)}_{2,i_{12}}\right).$$

The array $\mathbf{U}^{(1)}_2$ of standard uncertainties associated with the principal measurand estimates is given by

$$\mathbf{U}_2^{(1)} = \begin{bmatrix} u\left(y_{2,1,1}^{(1)}\right) & \dots & u\left(y_{2,n_\mathrm{S}^{(1)},1}^{(1)}\right) \\ \vdots & \ddots & \vdots \\ u\left(y_{2,1,m_2^{(2)}}^{(1)}\right) & \dots & u\left(y_{2,n_\mathrm{S}^{(1)},m_2^{(2)}}^{(1)}\right) \end{bmatrix},$$

where

$$u^2\left(y_{2,l_\mathrm{S},i_{12}}^{(1)}\right) = \mathbf{V}_{i_{12}}^{(1)}(l_\mathrm{S}, l_\mathrm{S}).$$